Review Session

1.1(b)

octahedral: the same quantities applies to cube, cube applies to the octahedral.

What to show: $\varphi: \mathrm{Sym}\,T \to S_4$.

$C = \{v_1 - v_1, v_2 - v_2, v_3 - v_3, v_4 - v_4\}$

$\Rightarrow$ choose the main diagonals.

$D = \{d_1, d_2, d_3, d_4\}$

$\varphi: \mathrm{Sym}\,C \to S_4$ → not full symmetry

$\langle (12)(1234) \rangle = \langle (12), (13), (14) \rangle$

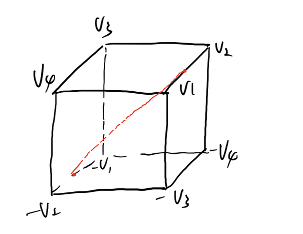
$\varphi: \mathrm{Sym}\,T \to S_4$.

if only rotation $\to A_4$.

if full symmetric $\to S_4$.

full symmetry - the product of decomposition [see online notes].

decomposition into decomposition, parity stays the same



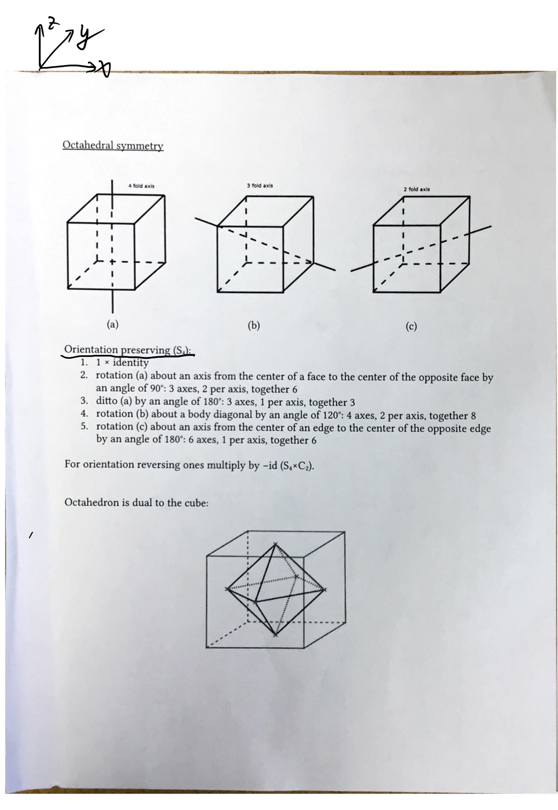
basis: $\{v_1, v_2, v_3\}$

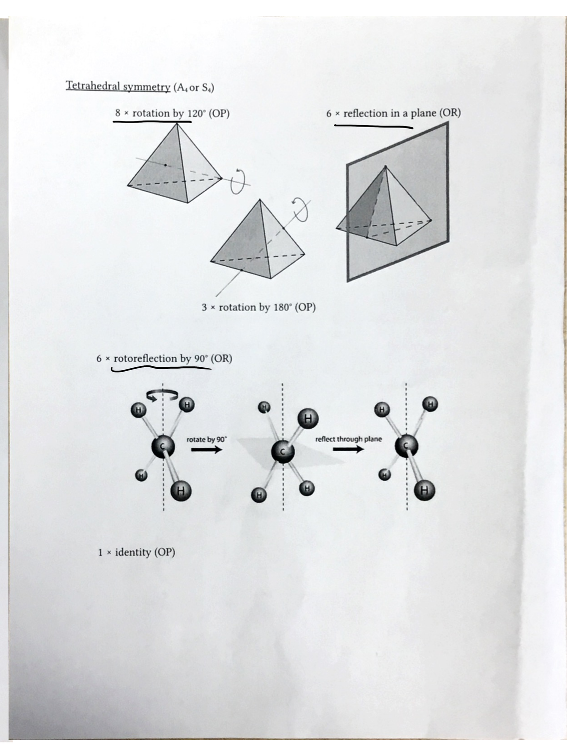
$v_1 \to v_2$, $v_2 \to v_1$, $v_3 \to -v_3$. $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

$v_1 = (1,1,1)$, $v_2 = (-1,1,1)$, $v_3 = (1,-1,1)$, $v_4 = (-1,-1,1)$

How to write a matrix rotation by $90°$ about z-axis. $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$

$P: G \to GL(\mathbb{C}^3)$ in the case of octahedral group $G \subset GL(\mathbb{C}^3)$, P is just the inclusion.





$C_n = \langle r \rangle$, $|r| = n$. cyclic group of order n.

Standard representation of C_n is the following matrices of the basis $\{e_1, e_2\}$.

$r^k \mapsto \begin{bmatrix} \cos\frac{2\pi k}{n} & -\sin\frac{2\pi k}{n} \\ \sin\frac{2\pi k}{n} & \cos\frac{2\pi k}{n} \end{bmatrix}$ irreducible as a real representation. no simultaneous eigenvector

eigenvalues $= e^{\frac{-2k\pi}{n}} \begin{bmatrix} 1 \\ i \end{bmatrix}$ → reducible over complex number.

$S \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$\begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \alpha \begin{bmatrix} a \\ b \end{bmatrix}$

$a\cos x - b\sin x = \alpha a$

$a(\cos x - \alpha) - b\sin x = 0$

$a = b$

$e^{ix} = \cos x + i\sin x$

Calculate the character and show $\langle \chi_p, \chi_p \rangle = 1$. then the representation is irreducible

$\frac{1}{|G|}\sum_{g \in G} |\chi(g)|^2$

$\chi(r^k) = 2\cos\frac{2\pi k}{n}$, $\chi(s) = 0$, $\chi(r^k s) = 0$

$r^k s \mapsto \begin{bmatrix} \cos\frac{2\pi k}{n} & 0 \\ 0 & -\cos\frac{2\pi k}{n} \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$? $\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rangle \oplus \langle \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rangle = \mathbb{C}^2$.

only two eigenspaces for the matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Decompose the standard representation of S_3 into irreducible ones.

$V = \mathbb{C}^3$ $\{e_1, e_2, e_3\}$. We can try to find a simultaneous eigenvector by the procedure

$(12) \mapsto \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ pick any $v \in V$. then define $w = \frac{1}{|G|}\sum_{g \in G} P(g)v$.

$\Rightarrow P(g)w = w$ for all $g \in G$. → an eigenvector. But w can be 0.

If we pick $v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow w = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

proving that an infinite group representation that is irreducible but could be decomposed.

Example: a group G + representation $P: G \to GL(V)$ such that
(1) P is reducible.
(2) P does not split into irreducible ones.

Decompose the standard representation of S_3 into irreducible ones.

$V = \mathbb{C}^3$, $\{e_1, e_2, e_3\}$ as basis.

Look at $W^\perp = \{(x_1, x_2, x_3), (1,1,1) = 0\}$

find $W^\perp = \{x_1 + x_2 + x_3 = 0\}$. — pick a basis for $W^\perp$ $\left\{ f_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, f_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}$

$(1) \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$(12)\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = -f_1$

$(12)\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = f_1 + f_2$

Let $P_{W^\perp}: G \to GL(\mathbb{C}^2)$ be the restriction of $P_s: G \to GL(\mathbb{C}^3)$.

then $\chi_{P_s}((1)) = 2$, $\chi_{P_s}((12)) = 0$, $\chi_{P_s}((123)) = \chi_P((123))$

$(123) f_1 = \dots = f_2 - f_1$

$(123) f_2 = \dots = -f_2$

$\chi_P((12)) = 1$

$\langle \chi_P, \chi_P \rangle = \frac{1}{|G|}\sum_{g \in G} |\chi_P(g)|^2 = \frac{1}{6}\left(2^2 + 3 \cdot 0^2 + 2 \cdot (-1) \cdot \overline{(-1)}\right) = 1$ ✓

What if some inner product doesn't satisfy?

We can produce one.

How: $\langle\langle v', v'' \rangle\rangle = \frac{1}{|G|}\sum_{g \in G} \langle P(g)v', P(g)v'' \rangle$

$G = P$, $P: G \to GL(\mathbb{C}^2)$ $k \mapsto \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$

[inside of M_2, x_2 ...]

We have $\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rangle$? $= \langle \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rangle$ invariant subspace.

need to decompose into two different invariant subspace

try to find another eigenvector.

$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix}$

$b = 0 \Rightarrow$ gives $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

or $\alpha = 1 \Rightarrow b = 0$.

G finite group, $P: G \to GL(V)$ a finite-dimensional $\mathbb{C}$-representation.

Assume there is an inner product on V, $\langle\,,\rangle$ such that $\langle P(g)v', P(g)v'' \rangle = \langle v', v'' \rangle$ for all $v', v'' \in V$ and $g \in G$.

[There always exist some kind of inner product, e.g. if we pick a basis $\{f_1, f_2, \dots, f_n\}$, then we can define $\langle (z_1, z_2, \dots, z_n), (w_1, w_2, \dots, w_n) \rangle = \sum_{i=1}^n z_i \overline{w_i}$]

Suppose inner product, we can do the following.

Let W be a P-invariant subspace of V.

We claim that $W^\perp$ (orthogonal complement with respect to inner product satisfying ?) is also P-invariant. And then we have the splitting $W \oplus W^\perp$ $\{w' : \langle w', w \rangle = 0 \ \forall w \in W\}$

proof: $w' \in W^\perp$, need to show that $P(g)w' \in W^\perp$ for all $g \in G$.

$\Leftrightarrow \langle P(g)w', w \rangle = 0$. $\forall w \in W$.

$\Leftrightarrow \langle w', P(g)^* w \rangle = 0$ $\forall w \in W$.

$\Leftrightarrow \langle w', P(g)^{-1} w \rangle = 0$ $\forall w \in W$,

W invariant. $\Rightarrow P(g)W \in W$.

$\Leftrightarrow \langle w', P(g)^{-1} w \rangle = 0$. ✓

$\langle P(g)v', P(g)v'' \rangle = \langle v', v'' \rangle \to$? $\Leftrightarrow P(g)$ n map an orthonormal basis to another orthonormal basis.

$\langle e_i, e_j \rangle = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j. \end{cases}$

$\langle P(g)^* P(g) v', v'' \rangle = \langle v', v'' \rangle$.

$\Rightarrow \langle (P(g)^* P(g) - I)v', v'' \rangle = 0$ $\forall v'' \in V$.

$\Rightarrow (P(g)^* P(g) - I)v' = 0$. $\Rightarrow P(g)^* P(g) v' = v'$. $\forall v' \in V$

$P(g)^* P(g) = I$ $P(g)^{-1} = P(g)^*$ adjoint.

Case: $S_3 \to GL(\mathbb{C}^3)$ is the usual inner product a nice one?

$\langle (z_1, z_2, z_3), (w_1, w_2, w_3) \rangle = z_1\overline{w_1} + z_2\overline{w_2} + z_3\overline{w_3}$

$P(g)$ $g \in S_3$. just permute $\{e_1, e_2, e_3\}$. $\{P(g)e_1, P(g)e_2, P(g)e_3\}$ is still orthonormal. Hence $\langle \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \rangle$ is invariant subspace, then its complement with respect to final inner product will be invariant.

D_4: character table $\{1, r, r^2, r^3, s, rs, r^2s, r^3s\}$

$g = r \Rightarrow g \cdot r^l g^{-1} = r^l$

$r^k s r^{-k} = \dots$

$g = r^k s \Rightarrow g \cdot r^l \cdot g^{-1} = r^k s r^l s r^{-k} = r^{-l}$

$g = r^k \Rightarrow r^k \cdot r^l s \cdot r^{-l} = r^{2l+k}s = r^{2l}r^k s$

$g = r^k s \Rightarrow r^k s \cdot r^l s \cdot (r^k s)^{-1} = r^l s r^k s s r^{-l} = r^l r^{-k} r^l s = r^{2l-k}s$.

	$\{1\}$	$\{r, r^3\}$	$\{r^2\}$	$\{s, r^2s\}$	$\{rs, r^3s\}$
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	-1	1	-1	1
χ_4	1	-1	1	1	-1
χ_5	2	-2	0	0	0

$\chi_i(1) = d_i$

$d_i \mid |G|$.

$|G| = d_1^2 + d_2^2 + \dots + d_k^2$

$8 = 1 + d_2^2 + d_3^2 + d_4^2 + d_5^2$
$\ 1 \quad 1 \quad 1 \quad 2$

$|\chi(g)| \leq d_i$

$P_i(g) = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_{d_i} \end{bmatrix}$ diagonal. $\xrightarrow{\mathrm{tr}} \lambda_1 + \dots + \lambda_d$.

$|\chi_i(g)| \leq |\lambda_1| + \dots + |\lambda_d| \leq d$.

$|\chi_1(g)| = d \Leftrightarrow P_1(g) = \lambda I$

$\chi_1(g) = d \Leftrightarrow P_1(g) = I$

★ if c will produce a new representation

Sanity Check: We know a 2D representation of D_4. (χ_5)

$e \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ 2

$r \to \begin{pmatrix} 0 & \\ & 0 \end{pmatrix}$ 0

$s \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ 0

$rs \to \begin{pmatrix} \cos\frac{2\pi}{4} & 0 \\ 0 & -\cos\frac{2\pi}{4} \end{pmatrix}$ 0

$r^2 \to \begin{pmatrix} -1 & \\ & -1 \end{pmatrix}$ -2.

If $\bar{P}: G/N \to GL(V)$ is irreducible. then $P := \bar{P} \circ \pi : G \xrightarrow{\pi} G/N \to GL(V)$ is irreducible.

Choose $\{1, r^2\}$ as normal subgroup $\Rightarrow$ [normal group is a union of conjugacy classes].

G/N has order 4 $\to$ not $\langle g \rangle$ since r & r has order 4. But r^2 is in $\{1, r^2\}$ → order 2.

$G/N \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

$r \mapsto (1,0)$
$s \mapsto (0,1)$
$rs \mapsto (1,1)$

	1	a $(1,0)$	b $(0,1)$	ab $(1,1)$
χ_1	1	1	1	1
χ_2	1	-1	1	-1
χ_3	1	1	-1	-1
χ_4	1	-1	-1	1

$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \xrightarrow{\pi_1} \mathbb{Z}/2\mathbb{Z}$.

$\{1, -1\} \times \{1, -1\} \xrightarrow{\pi_2} \{1, -1\} \subseteq \mathbb{C}^\times$. $\pi_1(a,b) = a$, $\pi_2(a,b) = b$.

a representation. homomorphism.

Let $P: G \to \mathbb{C}^\times$ be a group homomorphism (1-dimensional representation).

Let $P: G \to GL(V)$ be a representation with character χ.

Then we can define a new representation.

$P'(g) = P(g) \cdot P(g)$.

1) if P is irreducible $\Rightarrow P'$ is irreducible.

and $\chi_{P'}(g) = \varphi(g)\chi_P(g)$

D_5	(1) 1	(2) $\{r, r^4\}$	(2) $\{r^2, r^3\}$	5 $\{s, rs, r^2s, r^3s, r^4s\}$
χ_1	1	1	1	1
χ_2	1	1	1	-1
χ_3	2	$w + \bar{w}$	$w^2 + \bar{w}^2$	0
χ_4	2			

Standard Rep: $r \to \begin{bmatrix} \cos\frac{2\pi}{5} & \\ & \cos\frac{2\pi}{5} \end{bmatrix}$ $w = e^{\frac{2\pi}{5}i}$ $\bar{w} = e^{-\frac{2\pi}{5}i}$

$r^2 \mapsto \begin{bmatrix} \cos\frac{4\pi}{5} & \\ & \cos\frac{4\pi}{5} \end{bmatrix}$

χ_2: sign representation.

D_6	(1) 1	(1) r^3	(2) r	(2) r^2	(3) rs	(3) rs
χ_1	1	1	1	1	1	1
χ_2	1	1	1	1	-1	-1
χ_3	1	-1	-1	1	-1	1
χ_4	1	-1	-1	1	1	-1
χ_5	2	2	-1	-1	0	0
χ_6	2	-2	1	-1	0	0

$N = \{1, r^3\}$.

$G/N \cong S_3$.

	(1) 1	(2) (123)	(3) (12)
χ_1	1	1	1
χ_2	1	1	-1
χ_3	2	-1	0

$1 \to 1$
$r^3 \to 1$
$r \to (123)$
$s \to (12)$

Standard D_6:

$r \to \begin{bmatrix} 1/2 & \\ 0 & 1/2 \end{bmatrix}$

$r^2 \to \begin{bmatrix} 1/2 & \\ & 1/2 \end{bmatrix}$

$r^3 \to \begin{pmatrix} -1 & \\ & -1 \end{pmatrix}$

$s \to \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}$

$rs \to \begin{bmatrix} \cos & \\ & -\cos \end{bmatrix}$

G — noncommutative group of order 21.

Sylow — $\exists$ normal subgroups of order 7. ($n_7 \mid 3$. and $n_7 \equiv 1 \pmod 7$) $\Rightarrow n_7 = 1$.)

Sylow 3-group $n_3 \mid 7$ and $n_3 \equiv 1 \bmod 3$. $\Rightarrow n_3 = 7$ ($n_3 = 7 \Rightarrow \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$).

	(1) 1	(3) α	(3) β	(7) a	(7) $b\,[a^2]$
χ_1	1	1	1	1	1
χ_2	1	1	1	w	w^2
χ_3	1	1	1	w^2	w
χ_4	3	p	q	0	0
χ_5	3	r	s	0	0

some cyclic group.

$\{1\}$

$gNg^{-1} \subseteq N$.

$\Updownarrow$

N is a union of conjugacy classes $\{1\}$ is a conjugacy class

$\Downarrow$

$\{b\}$

Cannot have $|\{x, \dots, x\}| = 6$. Since conjugacy class has order $\mid 21$.

cannot have. $\{x\}, \{x\}, \{x\}, \{x\}, \{x\}, \{1\}$ are then N_7 is center & and $G/N \cong \mathbb{Z}/3\mathbb{Z}$ implies that G is abelian.

Cannot have $\{x, x, x\}$, $\{x\}$, $\{x\}$, $\{x\}$, $\Rightarrow |Z(G)| \geq 4$. $\Rightarrow Z(G) = N$ ✗ or $Z(G) = G$ ✗.

$N = \{1\}\ \{x', x'', x'''\}\ \{y', y'', y'''\}$.

by Sylow 2-theorem, all Sylow 3-group are conjugate of each other.

$1 + d_2^2 + d_3^2 + d_4^2 + d_5^2 = 21$.
$\ 1 \quad 1 \quad 3 \quad 3$

$G \to G/N_7 \cong \mathbb{Z}/3\mathbb{Z}$. N_7 normal subgroup α, β in it.

$\mathbb{Z}/3\mathbb{Z}$	(1) 0	(1) 1	(1) 2
χ_1	1	1	1
χ_2	1	w	w^2
χ_3	1	w^2	w

$\varphi(k) = e^{\frac{i2\pi k}{3}}$

$w = e^{2\pi i/3}$

p, q, r, s are all sum of 7 root of unity.

Sum of diagonal.

$\Downarrow$.

orthogonality: $3 + 3\bar{p} + 3\bar{q} = 0$.

$p + r = -1$

$q + s = -1$

$\langle \chi_4, \chi_4 \rangle = \frac{1}{21}\left(9 + 3|r^2|^2 + 3|q|^2\right) = 1$

$\langle \chi_4, \chi_5 \rangle = \frac{1}{21}\left(9 + 3p\bar{r} + 3q\bar{r}\right) \stackrel{?}{=} 0$

Now we examine the conjugacy class of the normal subgroup ⑦.

$gxg^{-1} = \alpha^e$. $N_7 = \langle a \rangle$

$g \in G$. $\to g \notin N$.

then g induces a function $\varphi: N_7 \to N_7$ $\varphi(x) = gxg^{-1}$, $\varphi \in \mathrm{Aut}(N_7) \cong \mathbb{Z}/6\mathbb{Z}$.

$\varphi_a(x) = ax$ $\mathbb{Z}/7\mathbb{Z}$

$\varphi_2 \quad \varphi_2^2 \quad \varphi_2^3 = id \to$ not enough

$\varphi_3 \to \varphi_3^2 \to \varphi_3^3 \to \varphi_3^4$.

a subgroup of order 6. $\mathbb{Z}/3\mathbb{Z} \longmapsto \mathrm{Aut}(N_7) \cong \mathbb{Z}/6\mathbb{Z}$.

$g \longmapsto (x \mapsto gxg^{-1})$.

$\mathbb{Z}/3\mathbb{Z} \longrightarrow \mathbb{Z}/6\mathbb{Z}$

$\{\alpha, \alpha^2, \alpha^4\}$ $\{\alpha^3, \alpha^6, \alpha^5\}$.

$P_4(P) = \begin{bmatrix} \lambda_1 & & \\ 0 & \lambda_2 & 0 \\ & & \lambda_3 \end{bmatrix}$ $\lambda_i = \rho$

$\chi_4(\alpha) = \chi_4(\alpha^2) = \chi_4(\alpha^4)$

Pick one λ

$P(\alpha)v = \lambda v$
$P(\alpha^2)v = \lambda^2 v$
$P(\alpha^4)v = \lambda^4 v$

$\rho = e^{\frac{2\pi i}{7}}$

$\begin{bmatrix} \lambda & & \\ & \lambda' & \\ & & \lambda'' \end{bmatrix}$ $\begin{bmatrix} \lambda^2 & & \\ & \lambda'^2 & \\ & & \lambda''^2 \end{bmatrix}$ $\begin{bmatrix} \lambda^4 & & \\ & \lambda'^4 & \\ & & \lambda''^4 \end{bmatrix}$

$\begin{bmatrix} \rho & & \\ & \rho^2 & \\ & & \rho^4 \end{bmatrix}$

$p = \rho$

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